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OPTIMIZATION OF PERIODIC SYSTEMS*

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The problem of designing a regulator, optimal by a quadratic performance criterion, on an infinite time interval is examined for a linear periodic system. It is assumed that the control plant's motion is described by a system of linear periodic finite-difference equations. Controllable plants whose motion is described by differential and by finite-difference equations on different parts of the period are analyzed as well. The optimal regulator design problem is reduced to the determination of a periodic solution of an appropriate Riccati equation. An algorithm for constructing such a solution is derived. It is noted that this result can be used in periodic optimization problems /1/ and in the design of a stabilization system for a pacing apparatus.

1. Let us consider the problem of an analytic design of a regulator for a discrete periodic system. Let the plant's motion be described by the system of finite-difference equations

$$\mathbf{x}(i+1) = \Psi(i)\,\mathbf{x}(i) + \Gamma(i)\,\mathbf{u}(i), \quad i = 1, 2, \ldots \tag{1.1}$$

For $x(0) \neq 0$ it is required to choose suitably a control strategy (the equation of the regulator)

$$\mathbf{u}(i) = f(\mathbf{x}(i)) \tag{1.2}$$

ensuring the stability of system (1.1) and (1.2) $(\lim_{i\to\infty} \mathbf{x}(i) = 0)$ and minimizing the quadratic performance index

$$I = \sum_{i=0}^{\infty} [\mathbf{x}'(i)\,Q(i)\,\mathbf{x}(i) + \mathbf{u}'(i)\,B(i)\,\mathbf{u}(i)] \tag{1.3}$$

Here $\mathbf{x}(i)$ and $\mathbf{u}(i)$ are the phase coordinate and the control vectors (the primes denote transposition), the matrices $\Psi(i)$, $\Gamma(i)$, $Q(i) = Q'(i) \geqslant 0$, $B(i) = B'(i) > 0$ are periodic with period p, i.e., $\Psi(i+p) = \Psi(i)$, $\Gamma(i+p) = \Gamma(i)$, etc.

It is well known (see /2/, for example) that for the given problem the optimal regulator's Eq.(1.2) has the following form:

$$\mathbf{u}(i) = -[\Gamma'(i)\,S(i+1)\,\Gamma(i) + B(i)]^{-1}\;\Gamma'(i)\;\;S(i+1)\times\Psi(i)\,\mathbf{x}(i) \tag{1.4}$$

The sequence of symmetric matrices S, defining the optimal control law, satisfies the recurrence relation

$$S(j) = \Psi'(j)\;[S(j+1) - S(j+1)\;\Gamma(j)\;(B(j) + \Gamma'(j)\;S(j+1)\;\Gamma(j))^{-1}\;\Gamma'(j) S(j+1)]\,\Psi(j) + Q(j) \tag{1.5}$$

Thus, to determine the control law (1.4) it is necessary to find the value of one element of the sequence of matrices $S(j)$ satisfying relation (1.5). As a rule, in similar problems (with a finite time interval $(i = 0, 1, 2, \ldots, n)$) the value of matrix $S(n)$ is indicated at the interval's right end. In the case at hand of an infinite time interval $(i = 0, 1, 2, \ldots)$ the determination of the value of matrix $S(j)$ for some value of index j is an independent problem.

As a consequence of the periodicity of the matrices occurring in the problem's conditions the control strategy is not changed if the reference origin is shifted by p indices $(i = p, p+1, \ldots)$. Therefore, the required matrix sequence also must satisfy a periodicity condition

$$S(j+p) = S(j) \tag{1.6}$$

To find the relation defining matrix $S(j)$ it is necessary to obtain an analog of Theorem 2 of /3/ (which shows how to express the solution of a matrix Riccati differential equation in terms of blocks of the transition matrix of the Euler differential equations of the corresponding variational problem) for the case when the plant's motion is described not by differential but by finite-difference equations. To the variational problem described by functional (1.3) and the coupling Eqs.(1.1) there corresponds a system of equations relating the variation of the phase vector $\mathbf{x}(i)$ and the conjugate variable vector $\lambda(i)$ (see /2/, for instance)

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$$\mathbf{x}(i+1) = \Psi(i)\,\mathbf{x}(i) - \Gamma(i)\,B^{-1}(i)\,\Gamma'(i)\lambda(i+1), \quad \lambda(i) = Q(i)\,\mathbf{x}(i) + \Psi'(i)\,\lambda(i+1)$$

We assume that $\Psi^{-1}(i)$ exists. Then

$$\left\|\begin{matrix}\mathbf{x}(i+1)\\ \lambda(i+1)\end{matrix}\right\| = A(i)\left\|\begin{matrix}\mathbf{x}(i)\\ \lambda(i)\end{matrix}\right\|, \quad A(i) = \left\|\begin{matrix}a_{11}(i) & a_{12}(i)\\ a_{21}(i) & a_{22}(i)\end{matrix}\right\| \tag{1.7}$$

$$a_{11}(i) = \Psi(i) + \Gamma(i)\,B^{-1}(i)\,\Gamma'(i)\,(\Psi'(i))^{-1}\,Q(i)$$

$$a_{12}(i) = -\Gamma(i)\,B^{-1}(i)\,\Gamma'(i)\,(\Psi'(i))^{-1}$$

$$a_{21}(i) = -(\Psi'(i))^{-1}\,Q(i), \quad a_{21}(i) = (\Psi'(i))^{-1}$$

$$\left\|\begin{matrix}\mathbf{x}(i+n)\\ \lambda(i+n)\end{matrix}\right\| = W^i(n)\left\|\begin{matrix}\mathbf{x}(i)\\ \lambda(i)\end{matrix}\right\|, \quad W^i(0) = E \tag{1.8}$$

$$W^i(n) = \left\|\begin{matrix}w_{11}{}^i(n) & w_{12}{}^i(n)\\ w_{21}{}^i(n) & w_{22}{}^i(n)\end{matrix}\right\| = A(i+n-1)\ldots A(i)$$

Here E is the unit matrix. Since $\lambda(i) = S(i)\,\mathbf{x}(i)$, according to (1.8) we have

$$[w_{21}{}^i(n) + w_{22}{}^i(n)\;S(i)]\;\mathbf{x}(i) = S(n+1)\;[w_{11}{}^i(n) + w_{12}{}^i(n)\;S(i)]\,\mathbf{x}(i)$$

Hence follows a representation of the solution of difference Eq.(1.5) in terms of blocks of the transition matrix of system (1.7) (a discrete analog of Theorem 2 of /3/)

$$S(n+i) = [w_{21}{}^i(n) + w_{22}{}^i(n)S(i)]\;[w_{11}{}^i(n) + w_{12}{}^i(n)\;S(i)]^{-1} \tag{1.9}$$

Notice the "asymmetry" of the expression defining a symmetric matrix S.

The matrix $A(i)$ defining system (1.7) satisfies the relation

$$A'(i)\,J'A(i)\,J = E, \quad J = \left\|\begin{matrix}0 & E\\ -E & 0\end{matrix}\right\|$$

Multiplying by J both sides of the recurrence relation defining the matrix $W^i(n)$ in (1.8), we obtain

$$W^i(n+1)\;J = A\;(i+n)\;W^i(n)\;J$$

Consequently

$$(W^i(n+1))'\;J'W^i(n+1)\;\;J = (W^i(0))'J'W^i(0)\;J = E$$

From this equality it follows that the blocks of matrix $W^i(n)$ satisfy the relations

$$w_{11}{}^i(n) = [(w_{22}{}^i(n))']^{-1}\;[E + (w_{12}{}^i(n))'\;\;w_{21}{}^i(n)] = [(w_{22}{}^i(n))']^{-1} + w_{12}{}^i(n)\;\;(w_{22}{}^i(n))^{-1}\,w_{21}{}^i(n) \tag{1.10}$$

$$[(w_{22}{}^i(n))']^{-1}\,(w_{21}{}^i(n))' = w_{21}{}^i(n)\;\;(w_{22}{}^i(n))^{-1} = U^i(n) \tag{1.11}$$

$$(w_{21}{}^i(n))'\;\;[(w_{22}{}^i(n))']^{-1} = (w_{22}{}^i(n))^{-1}\;\;w_{22}{}^i(n) = R^i(n) \tag{1.12}$$

We symmetrize (1.9) with the aid of (1.10)–(1.12). From (1.9) and (1.10) we obtain

$$S(i) = (w_{22}{}^i(n))^{-1}\;\;[E - S(n+i)\;\;U^i(n)]^{-1}\,S(n) + i) \times [(w_{22}{}^i(n))']^{-1} - R^i(n) \tag{1.13}$$

Since according to (1.11) the matrix $U^i(n)$ is symmetric, we represent it as a product of three matrices $((N^i(n))^{-1}$ exists)

$$U^i(n) = H^i(n)\,N^i(n)\;\;(H^i(n))', \quad (N^i(n))' = N^i(n)$$

We now write (1.9) in the symmetric form analogous to (1.5)

$$S(i) = \Phi^i(n)\;\;\{S(i+n) - S(i+n)\,H^i(n)\,[(H^i(n))'S(i+n) \times H^i(n) - (N^i(n))^{-1}]^{-1}\;\;(H^i(n))'\;\;S(i+n)\}\,(\Phi^i(n))' - R^i(n)\;\;(\Phi^i(n) = (w_{22}{}^i(n))^{-1}) \tag{1.14}$$

According to (1.7) and (1.8), the matrices occuring in this expression are determined by the following recurrence relations (in which $\Psi^{-1}(n)$ does not enter):

$$\Phi^i(n+1) = \Phi^i(n)\;\;(E - Q(n+i)\;\;U^i(n))^{-1}\;\;\Psi'\;\;(n+i)$$

$$\Phi^i(0) = E, \;\; U^i(n+1) = \Psi(n+i)\,U^i(n)\;\;(E - Q(n+i) \times U^i(n))^{-1}\;\;\Psi'(n+i) - \Gamma\;\;(n+i)\;\;B^{-1}\;\;(n+i)\;\;\Gamma'(n+i), \quad U^i(0) = 0$$

$$R^i(n+i) = R^i(n) - \Phi^i(n)\;\;(E - Q(n+i)\;\;U^i(n))^{-1} \times Q(n+1)\;\;(\Phi^i(n))', \;\; R^i(0) = 0$$

Together with the use of periodicity condition (1.6) expression (1.4) enables us to write a discrete algebraic Riccati equation which the desired matrix $S(i)$ satisfies, in the form

$$S(i) = \Phi^i(p)\ \{S(i) - S(i)\ H^i(p)\ [(H^i(p))' S(i) H^i(p) - (N^i(p))^{-1}]^{-1}\ (H^i(p))'\ S(i)\}\ (\Phi^i(p))' - R^i(p) \tag{1.15}$$

Thus, the problem of choosing matrix $S(i)$ which together with (1.5) determines the required periodic sequence (respectively, the control strategy (1.4)) is reduced to the problem of choosing a solution of Eq.(1.15). To make such a choice we take advantage of the following property of asymptotic stability of the optimal closed-loop system "plant plus regulator". If a solution of Eq.(1.15) has been found, such that the eigenvalues of the matrix

$$[E - U^i(p)\ S(p+1)]^{-1}\ (\Phi^i(p))' \tag{1.16}$$

which determines the variation of the phase vector of the closed-loop system "plant plus regulator" corresponding to the discrete algebraic Riccati Eq.(1.15) /2/, lie inside the unit circle (if such a solution exists, then, as a rule, it is unique and can be found, for example, by means of an algorithm /4/), then the system of Eqs,(1.1)—(1.4) is asymptotically stable and, consequently, this value of $S(i)$ determines the required periodic sequence. Let us show this.

Bearing in mind the periodicity of the matrices occurring in the conditions of the problem being examined, to prove the asymptotic stability of system (1.1)—(1.4) it is enough to show that the matrix stipulated by this system, connecting the vectors $\mathbf{x}(i+p)$ and $\mathbf{x}(i)$, (this matrix determines the variation of the phase vector over one period) has eigenvalues lying inside the unit circle. According to (1.8)

$$\mathbf{x}(i+p) = (w_{11}{}^i(p) + w_{12}{}^i(p)\ S(i))\ \mathbf{x}(i)$$

Making use of (1.10) and (1.13), we obtain

$$\mathbf{x}(i+p) = [E - w_{12}{}^i(p)\ (w_{22}{}^i(p))^{-1}\ S(p+i)]^{-1}\ [(w_{22}{}^i(p))']^{-1} \times \mathbf{x}(i)$$

The matrix connecting the vectors $\mathbf{x}(i+p)$ and $\mathbf{x}(i)$ in this recurrence relation coincides with (1.16). Therefore, if a solution of Eq.(1.15) has been chosen such that the absolute values of the eigenvalues of matrix (1.16) are less than one, then system (1.1)—(1.4) is asymptotically stable.

2. Let us consider the problem of designing analytically a regulator for a periodic controllable system whose motion is described by differential and by finite-difference equations on different parts of the period. An example of such a control object is a two-legged pacing apparatus.

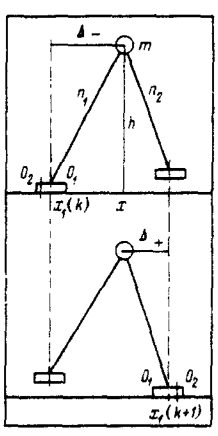

Fig.1

Let us illustrate the necessity of a different method of describing the pacing apparatus on different phases of its motion. Let the pacing apparatus be idealized as an inverted mathematical pendulum equipped with feet (Fig.1), consisting of a point mass m and two weightless legs n_1 and n_2 on which it supports itself in turn after a specified time interval τ (the mechanism accomplishes a one-pace regular walk). At the end of each leg there is a foot, viz., a device which by applying a moment at the point of junction of leg and foot (point O_1).

We assume that the foot's mass can be neglected. Neglecting as well the viscous forces, the equation of motion of such a system along the axis during the time it is supported on one leg is written as /5/

$$x^{\cdot\cdot} = gh^{-1}(x - x_2)$$

Here g is the gravitational acceleration, x is the coordinate of mass m, x_2 is the coordinate of point O_2. It is assumed that this coordinate can change during the pace because of a change in the moment applied at point O_1 (a continuous control). We place the origin of the coordinate system at point O_1. In a new (changing with pace number) coordinate system the mechanism's equation of motion becomes

$$(x^\circ)^{\cdot\cdot} = gh^{-1}x^\circ + gh^{-1}u,\ (k-1)\tau < t < k\tau,\ k = 1, 2, \ldots, \quad x^\circ = x - x_1(k),\ u = x_1(k) - x_2 \tag{2.1}$$

Here $x_1(k)$ is the coordinate of point O_1, which remains constant during the k-th pace being examined $((k-1)\tau < t < k\tau)$ and changes by a jump at the next pace (at the instant of change of support leg) of mangitude $v(k) = x_1(k+1) - x_1(k)$. Because of the assumption that the leg being moved has no inertia, the magnitude $v(k)$ can be chosen arbitrarily (a pulsed control).

At the instant of change of support leg $(t = k\tau)$ the horizontal velocity of the mass is continuous; therefore, its magnitude is the same in the old and in the new coordinate systems (the origins of these coordinate systems coincide with the leg's support points at the k-th and $(k+1)$st paces, respectively. Consequently,

$$(x^\circ)^\cdot (k\tau + 0) = (x^\circ)^\cdot (k\tau - 0) \qquad (2.2)$$

At the instant $t = k\tau$ the coordinate x° suffers a discontinuity since $\Delta_- = x^\circ (k\tau - 0)$ is measured in the old coordinate system, while $\Delta_+ = x^\circ (k\tau + 0)$, in the new. This is shown in Fig.1. In the upper part the mechanism is shown at the end of the k-th step (it is supported on leg n_1); in the lower, at the start of the $(k+1)$ st step (leg n_2 is the supporting one). We obtain

$$x^\circ (k\tau + 0) = x^\circ (k\tau - 0) - v(k) \qquad (2.3)$$

Consequently, the pacing apparatus with weightless legs being considered is described as a control plant by the differential Eqs. (2.1) when $(k-1)\times \tau < t < k\tau$ and by the difference Eqs. (2.2)—(2.3) when $t = k\tau\ (k = 1, 2, \ldots)$.

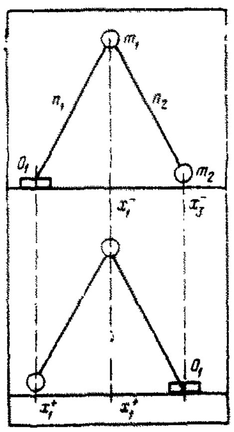

Fig.2

Let us complicate the pacing apparatus's model considered above by taking into account the mass of the foot on the moving leg (a model with weighted legs). Let the pacing apparatus be modelled as a double mathematical pendulum, shown in Fig.2, where m_1 is the mass of the mechanism's body, m_2 is the mass of the foot on the leg being moved, n_1 and n_2 are the mechanism's legs. Point O_1 is the junction of the leg and foot and serves as the origin of the coordinate system. We assume that besides the moment μ_1 acting in the foot a control moment μ_2 is applied to the leg being moved at the point it joins the body's mass. As in the previous example, we shall not examine the question of vertical stabilization of the mechanism, assuming that mass m_1 moves horizontally along the x-axis. We introduce the following phase coordinates of the mechanism: x_1 is the coordinate of mass $m_1, x_2 = x_1^\cdot, x_3$ is the coordinate of mass $m_2, x_4 = x_3^\cdot$ (Fig.2). In a neighborhood of the vertical position of the mechanism's body there is a domain of values of the phase coordinates (which are the components of vector $\mathbf{x}' = \| x_1, x_2, x_3, x_4 \|$ such that during the k-th pace (the period the mechanism is supported on one leg) the variation of vector $\mathbf{x}$ can be described with sufficient accuracy by a linear system of differential equations, in which the control vector has the form $\mathbf{u}' = \| \mu_1, \mu_2 \|$. At the instant of change of support leg $(t = k\tau)$ a jumpwise change occurs in the mechanism's phase coordinates. Let us find the relations describing the variation of the phase vector at the instant of change of support leg. Since origin of the reference system of the mechanism's phase vector coincides with the point of junction of support leg and foot, as seen on Fig.2 (the upper part of the figure shows the mechanism at the instant $t = k\tau - 0$, the lower, at $t = k\tau + 0$) the variations of coordinates x_1 and x_2 under change of support foot satisfy the following relations:

$$x_1 (k\tau + 0) = x_1 (k\tau - 0) - x_3 (k\tau - 0), \quad x_3 (k\tau + 0) = -x_3 (k\tau - 0) \qquad (2.4)$$

Assuming the boundedness of the moments μ_1 and μ_2, we obtain two more conditions

$$x_2 (k\tau + 0) = x_2 (k\tau - 0), \quad x_4 (k\tau + 0) = 0 \qquad (2.5)$$

We write relations (2.4) and (2.5) as

$$\mathbf{x} (k\tau + 0) = F_\delta \mathbf{x} (k\tau - 0), \quad F_\delta = \begin{Vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}$$

An analysis of these examples demonstrates the difference in the mathematical models of phase coordinate variation of pacing apparatuses with weighted and weightless legs.

Let us now analyze the general problem. On time intervals $(k-1)\tau < t < k\tau\ (k = 1, 2, \ldots)$ let a control plant's motion be described by a system of ordinary differential equations

$$\mathbf{x}^\cdot = F\mathbf{x} + G\mathbf{u} \qquad (2.6)$$

At the instant $t = k\tau$ the phase vector variation is subject to the law:

$$\mathbf{x} (k\tau + 0) = F_\delta \mathbf{x} (k\tau - 0) + M\mathbf{v}(k) \qquad (2.7)$$

We are required to find a strategy of continuous and pulsed controls $(\mathbf{u}(t) = f(\mathbf{x}(t)),\ \mathbf{v}(k) = \varphi(\mathbf{x}(k\tau - 0)))$ such that the closed-loop system "plant plus regulator" is asymptoically stable and such that this strategy minimizes the following quadratic functional (the performance index):

$$I(t_0) = \int_{t_0}^{\infty} (\mathbf{x}'Q\mathbf{x} + \mathbf{u}'B\mathbf{u})\, dt + \sum_{k=1}^{\infty} \mathbf{v}'(k) C \mathbf{v}(k) \qquad (2.8)$$

Here $F, G, B = B' > 0, Q = Q' > 0$ are periodic in t with period τ, the matrices $F_\delta, M, C = C' > 0$ are constant. Using the procedure, usual for linear quadratic Gaussian problems, of seeking the minimum of functional (2.8) as a quadratic form (see /2/, for example)

$$\min_{\mathbf{u}, \mathbf{v}} I(t_0) = \mathbf{x}'(t_0)\, S(t_0)\, \mathbf{x}(t_0)$$

we find

$$(k-1)\tau < t < k\tau, \quad \mathbf{u} = -B^{-1} G' S\mathbf{x} \tag{2.9}$$

$$t = k\tau, \quad \mathbf{v}(k) = -(M'S(k\tau - 0)\ M + C)^{-1}\ M'S(k\tau + 0) F_\delta\, x(k\tau - 0) \tag{2.10}$$

For $(k-1)\tau < t < k\tau$ the matrix S satisfies the differential Riccati equation

$$-S^{\cdot} = SF + F'S - SGB^{-1}G'S + Q \tag{2.11}$$

The jumps in this matrix at instant $t = k\tau$ are described by

$$S(k\tau - 0) = F_\delta' \{S(k\tau + 0) - S(k\tau + 0) M (C + M'S(k\tau + 0)M)^{-1} M'S(k\tau + 0)\} F_\delta \tag{2.12}$$

The invariance of the control strategy under a shift in the time reference origin by a period τ leads to a periodicity condition for matrix S

$$S(k\tau + 0) = S((k+1)\tau + 0), \quad k = 0, 1, 2, \ldots \tag{2.13}$$

Thus, as in the previous problem, to determine the optimal control strategy we need to find a periodic (with period τ) matrix S satisfying Eqs.(2.11), (2.12) and ensuring the asymptotic stability of system (2.6), (2.7), (2.9), (2.10). This problem reduces to the one considered in Sect.1 if the connection between $S(k\tau + 0)$ and $S((k+1)\tau - 0)$ is described not by differential Eq.(2.11) but by a relation analogous to (1.5). Let us find this relation. As follows from /3/, if $S(t)$ satisfies differential Eq.(2.11), then

$$S(t) = [\theta_{21}(t) + \theta_{22}(t) S(+0)]\ [\theta_{11}(t) + \theta_{12}(t)\ S(+0)]^{-1} \tag{2.14}$$

The matrices $\theta_{ij}(t)\ (i, j = 1, 2)$ are determined from the solution of the problem

$$\Phi^{\cdot} = \begin{Vmatrix} \theta_{11}^{\cdot} & \theta_{12}^{\cdot} \\ \theta_{21}^{\cdot} & \theta_{22}^{\cdot} \end{Vmatrix} = \begin{Vmatrix} F & -GB^{-1}G' \\ -Q & -F' \end{Vmatrix} \begin{Vmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{Vmatrix}, \quad \Phi(0) = E \tag{2.15}$$

As in Sect.1 let

$$J = \begin{Vmatrix} 0 & E \\ -E & 0 \end{Vmatrix}$$

Then

$$J\Phi(t) J'\Phi'(t) = \begin{Vmatrix} \theta_{22}\theta_{11}' - \theta_{21}\theta_{22}' & \theta_{22}\theta_{21}' - \theta_{21}\theta_{22}' \\ \theta_{11}\theta_{12}' - \theta_{12}\theta_{11}' & \theta_{11}\theta_{22}' - \theta_{12}\theta_{21}' \end{Vmatrix} = E \tag{2.16}$$

Relation (2.16) permits us to write (2.14) as

$$\begin{aligned} S(+0) = \theta_{22}^{-1}(t)\ [S(t) - S(t)\ D(t)\ (D'(t)\ S(t)\ D(t) - \\ N^{-1}(t))^{-1} D'(t)\ S(t)]\ (Q_{22}'(t))^{-1} - Q_{22}^{-1}(t)\ Q_{21}(t) \end{aligned} \tag{2.17}$$

Here the matrices $D(t), N(t)\ (N^{-1}(t)$ exists, $N'(t) = N(t))$ are determined by factoring the symmetric matrix $Q_{12}(t)\, Q_{22}^{-1}(t)$

$$D(t)\, N(t)\ D'(t) = Q_{12}(t)\ Q_{22}^{-1}(t) \tag{2.18}$$

The symmetry of matrices $U(t) = \theta_{12}(t)\, \theta_{22}^{-1}(t)$ and $R(t) = \theta_{22}^{-1}(t)\ Q_{21}(t)$ follows from (2.16); according to (2.15) they satisfy the symmetric differential equations and initial conditions

$$U^{\cdot} = FU + UF' + UQU - GB^{-1}G', \quad U(0) = 0 \tag{2.19}$$

$$R^{\cdot} = -\theta_{22}^{-1} Q (\theta_{22}^{-1})', \quad R(0) = 0 \tag{2.20}$$

We supplement these equations by one more, describing the variation of matrix θ_{22}^{-1}:

$$d\theta_{22}^{-1}/\,dt = \theta_{22}^{-1}(QU + F'), \quad \theta_{22}^{-1}(0) = E \tag{2.21}$$

Thus, the connection between $S(+0)$ and $S(\tau - 0)$ is completely determined from (2.17)–(2.21). We use these relations to solve the problem posed. Condition (2.13) (when $k = 0$) leads to an equation analogous to (1.15) in the matrix $S(+0)$

$$S(+0) = \theta_{22}^{-1}(\tau) F_\delta' [S(+0) - S(+0) K (\Pi^{-1} + K'S(+0)K)^{-1} K'S(+0)] F\ (\theta_{22}^{-1}(\tau))' - R(\tau) \tag{2.22}$$

The matrices Π, K are a result of factoring a symmetric matrix

$$(MC^{-1}M' - F_\delta U(\tau) F_\delta') = K\Pi K', \quad \Pi' = \Pi$$

Here the matrix Π^{-1} exists. As in the case of the discrete system considered in Sect.1, it can be proved that the required value of $S(+0)$ is that solution of Eq.(2.22) for which the matrix

$$[E + (MC^{-1}M' - F_\delta U(r)F_\delta')\ S(+0)]^{-1}\ F_\delta\,(\theta_{22}^{1-}(\tau))'$$

has eivenvalues lying inside the unit circle. In a special case, if $M = 0, F_\delta = E$ in (2.7), the value of $S(+0)$ found by using the algorithm /4/ from (2.22) determines a peridic solution of the matrix differential Riccati equation arising in problems of periodic optimization /1/. If matrix $Q = 0$ in (2.8), then, according to (2.20), $R(\tau) = 0$ and Eq.(2.22) can be transformed into a Liapunov equation. This case has been treated in detail (See: Larin, V.B., Stabilization of the horizontal motion of a two-legged pacing apparatus. Preprint No.4. Izd. Inst. Mat. Akad. Nauk SSSR, 1977).

REFERENCES

1. BITTANTI S., LOCATELLI A., and MAFFEZZONI C. Second-variation methods in periodic optimization. J. Optimizat. Theory and Appl.. Vol.14, No.1, pp.31-49, 1974.

2. BRYSON, Jr., A.E. and HO, Y.-C., Applied Optimal Control: Optimization, Estimation, and Control. Waltham, MA, Baisdell Publ. Co., 1969.

3. KALMAN R.E., and BUCY R.S. New results in linear filtering and prediction theory. Trans. ASME. Ser. D.J. Basis Eng., Vol.83, No.1, 1961.

4. ALIEV F.A., On the solution of a discrete algebraic Riccati equation. In: Cybernetics and Computation Engineering, Issue 38, Kiev, "Naukova Dumka", 1977.

5. LARIN V.B., On the continuous and pulsed control of the horizontal motion of a two-legged pacing apparatus. Izv. Akad.Nauk SSSR, Mekh. Tverd. Tela, No.6, 1977.

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